Journal of Statistical Physics, Vol. 63, Nos. 1/2, 1991

On the Convection of a Passive Scalar by a Turbulent Gaussian Velocity Field

T. C. Lipscombe,[1,2,3] **A. L. Frenkel,**[1,4] **and D. ter Haar**[2,5]

Received April 13, 1990; December 13, 1990

Through the use of the Novikov–Furutsu formula for Gaussian processes an equation is obtained for the diffusion of the ensemble average of a passive scalar in an incompressible turbulent velocity field in terms of the two-point, two-time correlator of this field. The equation is valid for turbulence which is not necessarily homogeneous or stationary and thus generalizes previous work.

KEY WORDS: Turbulence; diffusion; helicity; convection; incompressible fluids.

One topic of mathematical and physical interest in turbulence theory is the description of the diffusion of a passive scalar $C(\mathbf{x}, t)$ by an incompressible velocity field $\mathbf{v}(\mathbf{x}, t)$ whose statistics are known.

We shall consider this problem for a scalar $C(\mathbf{x}, t)$ which obeys the convection-diffusion equation

$$\frac{\partial C}{\partial t} + (\mathbf{v} \cdot \nabla)\, C = D\nabla^2 C \tag{1}$$

where D is the molecular diffusivity and the fluid has been assumed to have uniform properties. This equation is of practical importance, as $C(\mathbf{x}, t)$ could, for example, be the concentration of a pollutant or of a dye within the fluid. Knowledge of the behavior of this equation will also serve as a

[1] B. Levich Institute for Physicochemical Hydrodynamics, City College, New York, New York 10031.

[2] Department of Theoretical Physics, Oxford University, Oxford OX1 3NP, Great Britain.

[3] Present address: Physical Review B, 1 Research Road, Ridge, New York 11961.

[4] Present address: Department of Mathematics, University of Alabama, Tuscaloosa, Alabama 35487.

[5] Present address: P. O. Box 10, Petworth, West Sussex, GU28 ORY, Great Britain.

basis for considering more complicated physical systems, such as fluid flow in which two chemically reacting compounds are present. An additional term specifying the reaction kinetics should then be given.

Although Eq. (1) is linear in $C(\mathbf{x}, t)$, it can only be solved if the velocity field $\mathbf{v}(\mathbf{x}, t)$ is known. However, if the flow is turbulent, $\mathbf{v}(\mathbf{x}, t)$ —which is a solution of the Navier–Stokes equation—is not known. Therefore, it is impossible to find $C(\mathbf{x}, t)$. Hence, one tries to find an equation for the mean concentration where all quantities occurring in the equation are expressed in terms of the statistical properties of the turbulent velocity field, which we assume to be known.

We can rewrite Eq. (1) in the form of an integral equation,

$$C(\mathbf{x}, t) = C(\mathbf{x}, 0) + \int G(\mathbf{x}, t \mid \mathbf{x}', t')(\mathbf{v}(\mathbf{x}', t') \cdot \nabla')\, C(\mathbf{x}', t')\, d^3\mathbf{x}'\, dt' \qquad (2)$$

provided the initial concentration $C(\mathbf{x}, 0)$ and the appropriate boundary conditions are specified. In Eq. (2), $G(\mathbf{x}, t \mid \mathbf{x}', t')$ is the Green function for the diffusion equation,

$$G(\mathbf{x}, t \mid \mathbf{x}', t') = \theta(t - t')[4D(t - t')]^{-3/2} \exp\{-(\mathbf{x} - \mathbf{x}')^2/4D(t - t')\} \qquad (3)$$

where $\theta(t - t')$ is the Heaviside step function.

Equation (2) is a linear stochastic integral equation for $C(\mathbf{x}, t)$ with random coefficients. We refer to van Kampen's review paper[(1)] for a general discussion of such equations.

Before turning to our derivation of an equation for the ensemble average $\langle C(\mathbf{x}, t) \rangle$ of the field C, we want to mention briefly some earlier work on the diffusion problem in a fluid. The first major discussion of this problem is the one given by Taylor.[(2)] By first considering a discrete random-walk problem such that there is a nonzero correlation between successive displacements and then generalizing it to a continuous process, Taylor was able to write down an equation for the mean displacement; it has the form of a modified diffusion equation with a diffusion coefficient which is expressed in terms of the velocity correlation function.

Krachnan[(3)] also considered a Laplacian description of the turbulent velocity field for both two- and three-dimensional, incompressible, stationary, homogeneous, and isotropic fluids. He treated the velocity field as a Gaussian variable. In both the two- and three-dimensional cases he considered two energy spectra $E(k)$, one sharply peaked and one with a Gaussian shape, both centered at the same wave number k_0 corresponding to a certain correlation length, $l \sim k_0^{-1}$, for the velocity field. The time-correlation function of the velocity field was taken to be exponentially decaying, $\propto \exp(-\frac{1}{2}\omega_0^2 t^2)$. Numerical simulations were carried out and compared with the numerical solution of the corresponding equations which had been derived by Roberts,[4] who had applied Kraichnan's direct-interaction approximation (DIA)[5] to the diffusion problem. If $\omega_0 \sim v_0/l$, the reciprocal of the eddy circulation time (v_0 is the root-mean-square velocity in any direction), he expected, following Taylor,[2] that, provided $\omega_0 t \gg 1$, the eddy diffusion will act like a random walk with mean free path l and thermal velocity v_0. As $t \to \infty$, one expects that the Lagrangian velocity covariance will tend to zero, the effective eddy diffusivity will tend to a finite limit, $\sim v_0 l$, and the dispersion will become proportional to t. The two simulations were in good agreement, although the velocity fields generated were chosen from a Gaussian ensemble of solenoidal vector fields which were, on average, isotropic but, in general, not solutions of the Navier–Stokes equation.

Analytical studies were also carried out by Saffman,[6] Zel'dovich,[7] and Phythian and Curtis.[8] Saffman[6] introduced a normally distributed vector function $\mathbf{A}$ such that

$$\langle A_i(\mathbf{x}, t)\, A_j(\mathbf{x}+\mathbf{r}, t+\tau)\rangle = \delta_{ij}\delta(\mathbf{r})\,\delta(\tau) \tag{4}$$

He expanded all variables as series in Wiener–Hermite functions, that is, statistically orthonormal combinations of the A_i, so that the diffusion equation for $C(\mathbf{x}, t)$ is transformed into an equation for the unknown expansion coefficients. Closure results from averaging and truncation of the series. The equation which Saffman obtains—to $\mathcal{O}(A)$—is (summation is implied over repeated indices)

$$\frac{\partial\langle C\rangle}{\partial t} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\int_0^t \langle C(\mathbf{x}, t')\rangle\langle v_i(\mathbf{x}, t)\, v_j(\mathbf{x}, t+t')\rangle\, dt' \tag{5}$$

which is consistent with Taylor's result. Saffman suggests that the advantage of his procedure is that one can systematically improve the approximation by shifting the truncation to higher orders.

By first considering diffusion in the simple velocity field $\mathbf{v} = (2u\cos ky\cos\omega t, 0, 0)$ and then using a Kolmogorov cascade of similar eddies, Zel'dovich[7] finds the effective diffusivity in terms of the molecular diffusivity. Phythian and Curtis[8] use field-theoretic diagram techniques to find an expression for the long-time effective diffusivity for a Gaussian, homogeneous, isotropic, and stationary velocity field.

Recently, Avallaneda and Majda[9] have developed a Stieltjes-integral representation for the large-scale effective diffusivity in the case of "frozen turbulence," that is, an ensemble of steady flows. This is rigorously proven to be valid for all Péclet numbers Pe($= VL/D$, where V is a characteristic velocity such as the root-mean-square velocity and L a characteristic length such as the correlation length). However, the question of how to use this method to address some physically interesting problems has not yet been answered.

In the work of Saffman,[6] Zel'dovich,[7] and Phythian and Curtis,[8] by analogy with the eddy viscosity, the notion of an eddy diffusivity, which is possibly a tensor, arises. The effect of the turbulent fluid motion is to "renormalize" the molecular diffusivity. It is of practical importance to determine a form of the eddy diffusivity so that the governing equation (1) may be replaced by a simpler description which is realistic enough to represent adequately the behavior of the physical system. The exact nature of the eddy diffusivity has not yet been determined. Whether it lessens the molecular diffusivity—"destructive interference"—irrespective of the particular details of the fluid flow is not known. It is the form of the turbulent diffusivity that we shall discuss here. We note that the asymptotic existence of the eddy diffusivity has been proven in a mathematically rigorous way in the work of Papanicolaou and collaborators (ref. 10 and references therein). Obtaining expressions in terms of the turbulent velocity statistics, however, remains a challenging problem.

In the present paper we use a technique first developed by Moiseev *et al.*,[11] who considered the evolution of velocity and density perturbations in a compressible fluid. We have earlier[12] used the same technique to examine velocity perturbations in an incompressible fluid and Moiseev *et al.*[13] have used it to discuss vortex formation in a convecting fluid.

Let us assume that for times $t < 0$, $C(\mathbf{x}, t)$ is everywhere zero. At some initial instant, say, $t = 0$, the same concentration $\langle C(\mathbf{x}, 0)\rangle$ is imposed on every single realization of the flow. At all subsequent times the total concentration may be expressed as

$$C(\mathbf{x}, t) = \langle C(\mathbf{x}, t)\rangle + C^{\mathrm{fl}}(\mathbf{x}, t) \tag{6}$$

where C^{fl} is the fluctuation in the concentration, so that by definition $\langle C^{\mathrm{fl}}(\mathbf{x}, t)\rangle \equiv 0$.

The total concentration obeys for $t > 0$ the diffusion equation (1). If the velocity field is taken to have no mean flow, then an equation for the mean concentration can be found by inserting Eq. (6) into Eq. (1) and averaging. It is

$$\frac{\partial\langle C\rangle}{\partial t} + \langle(\mathbf{v}\cdot\nabla)\, C^{\mathrm{fl}}\rangle = D\nabla^2\langle C\rangle \tag{7}$$

The fluctuation $C^{\mathrm{fl}}(\mathbf{x}, t)$ is formally a functional of the turbulent velocity field. It may therefore be expanded in a functional Taylor series. Multiplying this expression by the velocity field and then averaging will give the correlator in Eq. (7) as a series in the moments of the velocity field. If the velocity field has a Gaussian distribution, then all even moments of $\mathbf{v}(\mathbf{x}, t)$

will reduce to the product of second-order moments, all odd moments being zero.

By this method the so-called Novikov–Furutsu[14,15] formula was derived. For any function $F(q)$ of a Gaussian field q of zero mean one has

$$\langle q(\mathbf{x}, t)\, F\rangle = \int \langle q(\mathbf{x}, t)\, q(\mathbf{x}', t')\rangle \left\langle \frac{\delta F}{\delta q(\mathbf{x}', t')}\right\rangle d^3\mathbf{x}'\, dt' \tag{8}$$

In our case, $F = C^{\mathrm{fl}}(\mathbf{x}, t)$ and $q(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t)$, so that the Novikov–Furutsu formula becomes

$$\langle v_i(\mathbf{x}, t)\, C^{\mathrm{fl}}(\mathbf{x}, t)\rangle = \int \langle v_i(\mathbf{x}, t)\, v_j(\mathbf{x}', t')\rangle \left\langle \frac{\delta C^{\mathrm{fl}}(\mathbf{x}, t)}{\delta v_j(\mathbf{x}', t')}\right\rangle d^3\mathbf{x}'\, dt' \tag{9}$$

The Gaussian assumption is used for mathematical simplicity. For the larger scales of turbulence, which can be imagined as independent eddies buffetted by the small-scale motion, this approximation is not too bad,[16] but experimental evidence suggests[17] that the small scales are not Gaussian, due to the fact that such scales are strongly intermittent.

In order to calculate the functional derivative required in Eq. (9), an equation is needed for $C^{\mathrm{fl}}(\mathbf{x}, t)$. This can be obtained by putting Eq. (6) into Eq. (1) and simplifying using Eq. (7); this gives

$$\frac{\partial C^{\mathrm{fl}}}{\partial t} + (\mathbf{v}\cdot\nabla)\langle C\rangle + (\mathbf{v}\cdot\nabla)\, C^{\mathrm{fl}} - \langle (\mathbf{v}\cdot\nabla)\, C^{\mathrm{fl}}\rangle = D\nabla^2 C^{\mathrm{fl}} \tag{10}$$

Using the Green function $G(\mathbf{x}, t\,|\,\mathbf{x}', t')$ for the diffusion equation, we can integrate Eq. (9) to give the formal solution

$$C^{\mathrm{fl}}(\mathbf{x}, t) = \int G(\mathbf{x}, t\,|\,\mathbf{x}', t)\{\langle (\mathbf{v}(\mathbf{x}', t')\cdot\nabla')\, C^{\mathrm{fl}}\rangle - (\mathbf{v}(\mathbf{x}', t')\cdot\nabla')\langle C\rangle - (\mathbf{v}(\mathbf{x}', t')\cdot\nabla')\, C^{\mathrm{fl}}\}\, d^3\mathbf{x}'\, dt' \tag{11}$$

where the initial condition $C^{\mathrm{fl}}(\mathbf{x}, 0) = 0$ has been used. Taking the functional derivative and then averaging, using

$$\frac{\delta v_i(\mathbf{x}, t)}{\delta v_j(\mathbf{x}', t')} = \delta_{ij}\delta(\mathbf{x} - \mathbf{x}')\, \delta(t - t')\, \theta(t - t') \tag{12}$$

we find

$$\left\langle \frac{\delta C^{\mathrm{fl}}(\mathbf{x}, t)}{\delta v_j(\mathbf{x}', t')}\right\rangle = -G(\mathbf{x}, t\,|\,\mathbf{x}', t')\frac{\partial \langle C(\mathbf{x}', t')\rangle}{\partial x_j'} - \int G(\mathbf{x}, t\,|\,\mathbf{x}'', t'')\left\langle v_k(\mathbf{x}'', t'')\frac{\partial}{\partial x_k''}\left[\frac{\delta C^{\mathrm{fl}}(\mathbf{x}'', t'')}{\delta v_j(\mathbf{x}', t')}\right]\right\rangle d^3\mathbf{x}''\, dt'' \tag{13}$$

This equation is not closed. If, however, the correlator

$$\left\langle v_k(\mathbf{x}'', t'') \frac{\partial}{\partial x_k''} \left[\frac{\delta C^{\mathrm{fl}}(\mathbf{x}'', t'')}{\delta v_j(\mathbf{x}', t')} \right] \right\rangle \tag{14}$$

is assumed to vary much more slowly in space than the Green function, we have

$$\begin{aligned} N &\equiv \int G(\mathbf{x}, t \mid \mathbf{x}'', t'') \left\langle v_k(\mathbf{x}'', t'') \frac{\partial}{\partial x_k''} \left[\frac{\delta C^{\mathrm{fl}}(\mathbf{x}'', t'')}{\delta v_j(\mathbf{x}', t')} \right] \right\rangle d^3\mathbf{x}''\, dt'' \\ &\approx \int dt'' \left\langle v_k(\mathbf{x}', t'') \frac{\partial}{\partial x_k'} \left[\frac{\delta C^{\mathrm{fl}}(\mathbf{x}', t'')}{\delta v_j(\mathbf{x}', t')} \right] \right\rangle \\ &\quad \times \int d^3\mathbf{x}''\, G(\mathbf{x}, t \mid \mathbf{x}'', t'') \end{aligned} \tag{15}$$

This will, for example, be the case when

$$\frac{(D\tau_c)^{1/2}}{L} \ll 1 \tag{16}$$

where L is the correlation length and τ_c the correlation time. The integral over space can be done analytically, if we use Eq. (3), and we get

$$N = \frac{\partial}{\partial x_k'} \int dt'' \left\langle v_k(\mathbf{x}', t'') \frac{\partial}{\partial x_k'} \left[\frac{\delta C^{\mathrm{fl}}(\mathbf{x}', t'')}{\delta v_j(\mathbf{x}', t')} \right] \right\rangle \theta(t - t') \tag{17}$$

This may be estimated as

$$N \approx \frac{v\tau_c}{L} \left\langle \frac{\delta C^{\mathrm{fl}}}{\delta v} \right\rangle \tag{18}$$

where v is a typical velocity, such as the root-mean-square velocity, which can be expressed in terms of the Péclet number as $v = \mathrm{Pe} \cdot D/L$. The contribution of the integral N in Eq. (13) is negligible as compared to the left-hand side provided $v\tau_c/L \ll 1$. Thus, the conditions for our theory to be valid can be written as

$$\frac{(D\tau_c)^{1/2}}{L} \ll 1 \qquad \text{and} \qquad \mathrm{Pe} \left[\frac{(D\tau_c)^{1/2}}{L} \right]^2 \ll 1 \tag{19}$$

Diffusive or convective effects may dominate, and the theory will still hold, provided the Péclet number is not too large. These two conditions can also be interpreted as saying that the correlation time must be small compared to the turnover and diffusion times. Given Eqs. (19), the "mixed" correlator is found to equal

$$\langle v_i(\mathbf{x}, t)\, C^{\mathrm{fl}}(\mathbf{x}, t)\rangle = -\int R_{ij}(\mathbf{x}, t \mid \mathbf{x}', t')\, G(\mathbf{x}, t \mid \mathbf{x}', t) \times \frac{\partial \langle C(\mathbf{x}', t)\rangle}{\partial x_j'}\, d^3\mathbf{x}'\, dt' \tag{20}$$

where R_{ij} is the two-point, two-time correlator of the turbulent velocity field,

$$R_{ij}(\mathbf{x}, t \mid \mathbf{x}', t') = \langle v_i(\mathbf{x}, t)\, v_j(\mathbf{x}', t')\rangle \tag{21}$$

Combining Eqs. (7) and (20) gives the equation which governs the evolution of the mean concentration:

$$\frac{\partial\langle C\rangle}{\partial t} = D\nabla^2\langle C\rangle + \frac{\partial}{\partial x_i}\int R_{ij}(\mathbf{x}, t \mid \mathbf{x}', t')\, G(\mathbf{x}, t \mid \mathbf{x}', t') \times \frac{\partial\langle C(\mathbf{x}', t')\rangle}{\partial x_j'}\, d^3\mathbf{x}'\, dt' \tag{22}$$

This equation is a generalization of Saffman's result. It is valid for the case when the molecular diffusivity is nonzero, and it is also valid for nonhomogeneous turbulence. Given the two-point, two-time velocity correlation function and the initial concentration $\langle C(\mathbf{x}, 0)\rangle$, it should be possible to find from Eq. (22) the concentration at any subsequent time, either analytically or numerically. The effect of the turbulent motion has been to produce a diffusivity tensor which is space- and time-dependent and which is not necessarily diagonal in form. The Green function, it should be remembered, has as a factor the Heaviside step function $\theta(t-t')$, ensuring that causality is not violated. As Eq. (22) is not restricted to homogeneous turbulence, the mean helicity density $\gamma = \langle(\mathbf{v}\cdot\boldsymbol{\omega})\rangle$ ($\boldsymbol{\omega}$ is the vorticity) is not necessarily zero, and may affect the value of the integral on the right-hand side. Numerical simulations by Drummond *et al.*[18] seem to suggest that existing theories describe nonhelical turbulence reasonably well, but fail when a small value for the helicity is introduced. Their "velocity fields" are generated in the same way, however, as in the simulations of Kraichnan.[3]

If $D=0$—the case in which the concentration field is "frozen" into the flow—we have

$$G(\mathbf{x}, t \mid \mathbf{x}', t') \to \delta(\mathbf{x}-\mathbf{x}') \tag{23}$$

As a consequence, integration over the spatial coordinates on the right-hand side of Eq. (22) can be carried out, the result being

$$\frac{\partial\langle C\rangle}{\partial t}=\frac{\partial^2}{\partial x_i\partial x_j}\int_0^t R_{ij}(\mathbf{x}, t\,|\,\mathbf{x}', t')\langle C(\mathbf{x}, t')\rangle\,dt' \tag{24}$$

which is precisely Saffman's result, Eq. (5), for the effective diffusivity when $D=0$.

If the flow is homogeneous, isotropic, and stationary, we have

$$R_{ij}(\mathbf{x}, t\,|\,\mathbf{x}', t')=\frac{\langle v^2\rangle}{3}\,\delta_{ij}\phi(t-t') \tag{25}$$

where $\phi(\tau)$ is a dimensionless time correlation function. This leads to the equation

$$\frac{\partial\langle C\rangle}{\partial t}=\frac{\langle v^2\rangle}{3}\nabla^2\int_0^t \phi(t-t')\langle C(t')\rangle\,dt' \tag{26}$$

If the correlation time is much shorter than any other relevant time scale, the time correlation function can be modeled as $\phi(\tau)=\tau_c\delta(\tau)$, so that

$$\frac{\partial\langle C\rangle}{\partial t}=\frac{\langle v^2\rangle}{6}\,\tau_c\nabla^2\langle C(t)\rangle \tag{27}$$

This is simply the diffusion equation; the eddy diffusivity is in this case positive definite. We note that a complementary problem, the determination of the structure function of a passive scalar, has been addressed in two recent papers, one by Effinger and Grossmann[19] and the other by Lesieur and Rogallo.[20] Taken together, the evolution and properties of the mean scalar field and its fluctuations provide a fuller description of turbulent flow.

We have thus derived an equation which governs the evolution of the mean concentration for the case of a turbulent velocity field that is incompressible, is of zero mean, and possesses a Gaussian distribution. The equation is valid for turbulence which is not necessarily homogeneous or stationary and it thus generalizes previous work. When $D=0$, the equation reduces to Saffman's result.

ACKNOWLEDGMENTS

Two of us (T.C.L. and A.L.F.) acknowledge with gratitude support by the U.S. Department of Energy, grant numbers DE-FG0288ER13822, DE-FG0288ER13837, and DFG05-90EK14100. One of us (A.L.F.) also acknowledges support from the National Science Foundation, grant number $R_2$8996152.

REFERENCES

1. N. G. van Kampen, *Phys. Rep.* **24**:171–228 (1976).
2. G. I. Taylor, *Proc. Lond. Math. Soc., Ser. 2* **20**:196–212 (1922).
3. R. H. Kraichnan, *Phys. Fluids* **13**:22–31 (1970).
4. P. H. Roberts, *J. Fluid Mech.* **11**:257–283 (1961).
5. R. H. Kraichnan, *J. Fluid Mech.* **5**:497–543 (1959).
6. P. G. Saffman, *Phys. Fluids* **12**:1786–1798 (1969).
7. Ya. B. Zel'dovich, *Sov. Phys. Dokl.* **27**:797–799 (1982).
8. R. Phythian and W. D. Curtis, *J. Fluid Mech.* **89**:241–250 (1978).
9. M. Avellaneda and A. J. Majda, *Phys. Rev. Lett.* **62**:753–755 (1989).
10. D. W. McLaughlin, G. C. Papanicolaou, and O. R. Pironneau, *SIAM J. Appl. Math.* **45**:780–797 (1985).
11. S. S. Moiseev, R. Z. Sagdeev, A. V. Tur, G. A. Khomenko, and V. V. Yanovskiĭ, *Sov. Phys. JETP* **58**:1149–1153 (1983).
12. T. Lipscombe, A. L. Frenkel, and D. ter Haar, *J. Stat. Phys.* **53**:95–108 (1988).
13. S. S. Moiseev, P. B. Rutkevich, A. V. Tur, and V. V. Yanovskĭ, *Sov. Phys. JETP* **67**:294–299 (1988).
14. E. A. Novikov, *Sov. Phys. JETP* **20**:1290–1294 (1965).
15. K. Furutsu, *J. Opt. Soc. Am.* **62**:240–254 (1972).
16. A. S. Monin and A. M. Yaglom, *Statistical Fluid Dynamics* (MIT Press, Cambridge Massachusetts, 1975), Vol. 2, §18.1.
17. C. Meneveau and K. R. Sreenivasan, *Nucl. Phys. B* **2**:49–76 (1987).
18. I. T. Drummond, S. Duane, and R. R. Horgan, *J. Fluid Mech.* **138**:75–91 (1984).
19. H. Effinger and S. Grossmann, *Phys. Fluids A* **1**:1021–1026 (1989).
20. M. Lesieur and R. Rogallo, *Phys. Fluids A* **1**:718–722 (1989).